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On Approximation by Translates of Globally Supported Functions

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We consider L_p-approximation ($1 \leqslant p \leqslant \infty$) from the dilates of a space generated by a finite number of functions that have mild polynomial decay at infinity. In particular, the local-controlled density order of such a family of approximating spaces is characterized in terms of the Strang–Fix condition.

1. INTRODUCTION

We are interested in approximation from dilates of a shift-invariant subspace $S = S(\Phi)$ of $L_p(\mathbb{R}^n)$ $(1 \geqslant p \leqslant \infty)$ generated by a finite family of functions $\Phi = \{\phi_1, ..., \phi_N\}$, which are not necessarily compactly supported but have a suitable decay rate at infinity. Our results extend recent work [14] and a result concerning density of multiresolutions in the paper of Jia and Micchelli [16].

The integer translates of a function on $\mathbb{R}$ were already considered by Schoenberg [18] in the 1940s. His work was extended by Strang and Fix [20], who characterized the controlled L_2-approximation order provided by integer translates of a compactly supported function in terms of the conditions that are now named the Strang–Fix conditions. For the more general case, where Φ consists of several compactly supported functions, de Boor and Jia [3] introduced the notion of local approximation and characterized the local approximation order in terms of the Strang–Fix conditions. See [1] for a systematical treatment of this issue.

The above problem is continuing to receive much attention. Several authors have investigated the approximation power provided by translates of a finite number of functions having global support (e.g., [4, 9, 10, 14, 17]), because the recent study with respect to interpolation by radial basis

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functions and the wavelet approximation require such consideration. In order to obtain a definite order of approximation, in the papers referred to above, it is assumed that functions ϕ have the decay rate $n+k+\lambda$ at ∞, for some $\lambda > -1$, namely,

$$|\phi(x)| = \mathcal{O}(\|x\|^{-(n+k+\lambda)}), \qquad \text{as} \quad x \to \infty. \tag{1.1}$$

Here we denote by $\|x\|$ the uniform norm of $x = (x_1, ..., x_n) \in \mathbb{R}^n$ given by $\max_{1 \leqslant j \leqslant n} |x_j|$.

Jia and Micchelli [16] considered the density of the multiresolutions generated by a function ϕ that satisfies a decay condition weaker than (1.1). To be precise, they introduced the subspace $\mathcal{L}_p$ of $L_p = L_p(\mathbb{R}^n)$ for $1 \leqslant p \leqslant \infty$ as follows: Given a function ϕ defined on $\mathbb{R}^n$, set

$$\phi^\circ := \sum_{j \in \mathbb{Z}^n} |\phi(\cdot - j)| \qquad \text{and} \qquad |\phi|_p := \|\phi^\circ\|_p\,([0, 1)^n). \tag{1.2}$$

Then $\mathcal{L}_p$ is defined to be the linear space of all functions ϕ for which $|\phi|_p < \infty$. Equipped with the norm $|\cdot|_p$, $\mathcal{L}_p$ becomes a Banach space. Clearly, $\|\phi\|_p \leqslant |\phi|_p$, and $|\phi|_p \leqslant |\phi|_q$ for $1 \leqslant p \leqslant q \leqslant \infty$. This shows that

$$\mathcal{L}_p \subseteq L_p \qquad \text{and} \qquad \mathcal{L}_q \subseteq \mathcal{L}_p, \qquad \text{for} \quad 1 \leqslant p \leqslant q \leqslant \infty.$$

THEOREM [16]. *If* $\phi \in \mathcal{L}_p$ $(1 \leqslant p < \infty)$ *and* $\sum_{j \in \mathbb{Z}^n} \phi(\cdot - j) = 1$, *then* ϕ *provides* L_p*-approximation of order at least* $o(1)$ *for the functions in* L_p.

An interesting observation concerning Jia and Micchelli's theorem is that if ϕ satisfies only the weaker decay condition $\phi \in \mathcal{L}_p$, then the approximation order guaranteed by the property $\sum_{j \in \mathbb{Z}^n} \phi(\cdot - j) = 1$ is only $o(1)$, not $\mathcal{O}(h)$ as one may expect. Motivated by this observation, we investigate in this paper the case where the decay condition (1.1) is weakened to be

$$(1 + \|\cdot\|)^{k-1}\,\phi \in \mathcal{L}_p, \tag{1.3}$$

where $1 \leqslant p \leqslant \infty$. We denote by $\mathcal{L}_p^{k-1}$ the space of the functions ϕ in $L_p(\mathbb{R}^n)$ having the property (1.3). After discussing the Strang–Fix conditions in Section 2, we show in Section 3 that $o(h^{k-1})$ is a lower bound of approximation provided by a finite number of functions in $\mathcal{L}_p^{k-1}$ satisfying the Strang–Fix conditions of order k. We also introduce in Section 4 the notion of the local-controlled approximation order $o(h^{k-1})$ and show that this order is already sufficient for the Strang–Fix conditions of order k.

In the rest of this section we introduce some notation to be used throughout the paper. Let $\mathbb{R}^n$ be the n-dimensional real linear space equipped with the uniform norm. Given $x, y \in \mathbb{R}^n$, we denote by $x \cdot y$ the inner product as usual.

We use the standard multi-index notation. Let $\mathbb{N}$ be the set of nonnegative integers. Given an element $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, the length of α is defined to be $|\alpha| := \sum_{j=1}^n \alpha_j$ and the factorial of α is $\alpha! := \alpha_1! \cdots \alpha_n!$. If $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ and $\beta = (\beta_1, ..., \beta_n) \in \mathbb{N}^n$, we say $\beta \leqslant \alpha$ provided $\beta_j \leqslant \alpha_j$ for $1 \leqslant j \leqslant n$. We also let $\binom{\alpha}{\beta} := \alpha!/\beta!\,(\alpha-\beta)!$ for $\beta \leqslant \alpha$.

Let $\mathbb{Z}$ be the set of integers. An element of $\mathbb{Z}^n$ is called a multi-integer. A mapping from $\mathbb{Z}^n$ to $\mathbb{C}$ is called a sequence on $\mathbb{Z}^n$. Given a sequence c we denote by $\|c\|_{l_p}$ the l_p-norm of c.

All the functions appearing in the paper are complex-valued and Lebesgue measurable. For $j = 1, ..., n$, we denote by $D_j f$ the partial derivative of a given function f with respect to the jth coordinate. In general, this derivative is in the distribution sense. For $\gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{N}^n$ the differential operator D^γ is defined by $D^\gamma := D_1^{\gamma_1} \cdots D_n^{\gamma_n}$. We denote by D_u the directional derivative given by $D_u = \sum_{j=1}^n u_j D_j$, for $u = (u_1, ..., u_n) \in \mathbb{R}^n$.

We denote by $\Pi = \Pi(\mathbb{R}^n)$ the linear space of all polynomials on $\mathbb{R}^n$ and by Π_k its subspace of all polynomials of (total) degree at most k, where k is a nonnegative integer. A mapping T on Π is called degree-reducing if for any $p \in \Pi$, Tp is a polynomial of degree less than the degree of p.

Given $k \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$, the Sobolev space W_p^k is the set of all functions f such that $\|f\|_{k,p} := \sum_{|\gamma| \leqslant k} \|D^\gamma f\|_p$ is finite. We denote by $|f|_{k,p}$ the semi-norm of $f \in W_p^k$ given by $|f|_{k,p} := \sum_{|\gamma| = k} \|D^\gamma f\|_p$. For a function $f \in L_1$, the Fourier transform of f is defined to be

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^n.$$

2. The Strang–Fix Conditions

The Strang–Fix conditions arise in the characterization of the approximation power provided by translates of functions, because these conditions yield certain polynomial reproduction properties. In this section we consider the Strang–Fix conditions and their equivalent forms for globally supported functions $\phi \in \mathscr{L}_1^{k-1}$, with no restriction on the continuity of these functions. As usual, the basic tool in the analysis of the Strang–Fix conditions is the Poisson summation formula, which may be formulated as follows.

Lemma 2.1. *Suppose ϕ is in $L_1(\mathbb{R}^n)$ and $\hat{\phi}(2\pi\cdot)|_{\mathbb{Z}^n}$ is in $l_1(\mathbb{Z}^n)$. Then*

$$\sum_{j \in \mathbb{Z}^n} \phi(x-j) = \sum_{j \in \mathbb{Z}^n} \hat{\phi}(2\pi j)\, e^{-i2\pi j\cdot x} \tag{2.1}$$

holds for almost every $x \in \mathbb{R}^n$.

Proof. Since $\phi \in L_1(\mathbb{R}^n)$, we have

$$\int_{[0,1)^n} \sum_{j \in \mathbb{Z}^n} |\phi(x-j)| \, dx = \int_{\mathbb{R}^n} |\phi(x)| \, dx < \infty.$$

Therefore the function

$$f(x) := \sum_{j \in \mathbb{Z}^n} \phi(x-j) \tag{2.2}$$

is defined for almost every $x \in [0, 1)^n$ and is 1-periodic. We refer to f as the periodization of ϕ. By Lebesgue's dominated convergence theorem, the Fourier coefficient $\hat{f}(j)$ for $j \in \mathbb{Z}^n$ is

$$\begin{aligned}\hat{f}(j) &:= \int_{[0,1)^n} f(x) \, e^{-i2\pi j \cdot x} \, dx \\ &= \sum_{\beta \in \mathbb{Z}^n} \int_{[0,1)^n} \phi(x-\beta) \, e^{-i2\pi j \cdot x} \, dx \\ &= \int_{\mathbb{R}^n} \phi(x) \, e^{-i2\pi j \cdot x} \, dx = \hat{\phi}(2\pi j).\end{aligned}$$

Thus f has the following Fourier series expansion on $[0, 1]^n$:

$$f(x) \sim \sum_{j \in \mathbb{Z}^n} \hat{\phi}(2\pi j) \, e^{i2\pi j \cdot x}. \tag{2.3}$$

Since $\hat{\phi}(2\pi \cdot)|_{\mathbb{Z}^n} \in l_1(\mathbb{Z}^n)$, the Fourier series in (2.3) converges absolutely for every $x \in \mathbb{R}^n$. The sum of this series equals f a.e. on $\mathbb{R}^n$ (see, e.g., [19, Chapter 7, Corollary 1.8]). This proves (2.1). ▌

Let us next consider the Strang–Fix conditions for one function. Let k be a positive integer. Recall that the linear space $\mathcal{L}_1^{k-1}$ consists of those functions ϕ on $\mathbb{R}^n$ for which the function given by $x \mapsto (1 + \|x\|)^{k-1} \phi(x)$ is in $L_1(\mathbb{R}^n)$. Let $\phi \in \mathcal{L}_1^{k-1}$. For any $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant k-1$, the function ϕ_α given by

$$\phi_\alpha(x) := (-x)^\alpha \, \phi(x), \qquad x \in \mathbb{R}^n,$$

is in $L_1(\mathbb{R}^n)$. Its Fourier transform is

$$\hat{\phi}_\alpha(\xi) = (-iD)^\alpha \, \hat{\phi}(\xi), \qquad \xi \in \mathbb{R}^n. \tag{2.4}$$

A function $\phi \in \mathcal{L}_1^{k-1}$ is said to satisfy the Strang–Fix conditions of order k, provided its Fourier transform satisfies the following conditions:

$$\hat{\phi}(0) = 1 \tag{2.5}$$

and

$$D^\alpha \hat{\phi}(2\pi j) = 0 \qquad \text{for all} \quad |\alpha| < k \text{ and } j \in \mathbb{Z}^n \setminus \{0\}. \tag{2.6}$$

Let c be a sequence on $\mathbb{Z}^n$ such that $|c(j)| \leqslant M(1 + \|j\|)^{k-1}$ for all $j \in \mathbb{Z}^n$, where M is a constant independent of j. In this case the sum $\sum_{j \in \mathbb{Z}^n} \phi(x-j)\, c(j)$ converges at almost every $x \in \mathbb{R}^n$, and the semi convolution of ϕ and c is defined to be

$$\phi *' c := \sum_{j \in \mathbb{Z}^n} \phi(\cdot - j)\, c(j).$$

THEOREM 2.2. *A function $\phi \in \mathscr{L}_1^{k-1}$ satisfies the Strang–Fix conditions of order k if and only if the mapping $1 - \phi *'$ is degree-reducing on Π_{k-1}.*

Proof. Suppose ϕ satisfies the Strang–Fix conditions of order k. Then by (2.4) and (2.6) we have

$$\hat{\phi}_\alpha(2\pi j) = 0 \qquad \text{for all} \quad j \in \mathbb{Z}^n \setminus \{0\} \text{ and } |\alpha| < k.$$

Let f_α be the periodization of ϕ_α:

$$f_\alpha := \sum_{j \in \mathbb{Z}^n} \phi_\alpha(\cdot - j).$$

Applying the Poisson summation formula, we find that f_α equals the constant $\hat{\phi}_\alpha(0)$ a.e. on $\mathbb{R}^n$. In particular, $f_0 = 1$ a.e. on $\mathbb{R}^n$, for $\hat{\phi}_0(0) = \hat{\phi}(0) = 1$ by (2.5). By the binomial theorem we have

$$j^\alpha = (x + (j + x))^\alpha = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} x^{\alpha - \beta} (j - x)^\beta.$$

It follows that

$$\phi *' (\ )^\alpha = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} x^{\alpha - \beta} \phi_\beta *' 1, \tag{2.7}$$

where $(\ )^\gamma$ stands for the monomial $x \mapsto x^\gamma$. But for $\beta \leqslant \alpha$ and $|\alpha| < k$,

$$\phi_\beta *' 1 = f_\beta = \hat{\phi}_\beta(0).$$

This together with (2.7) implies that $(1 - \phi *')(\ )^\alpha$ is a polynomial of degree $< |\alpha|$, as desired.

Suppose conversely that the mapping $1 - \phi *'$ is degree-reducing on Π_{k-1}. We show that ϕ satisfies the Strang–Fix conditions of order k. Let $|\alpha| < k$. Using the binomial theorem again, we obtain

$$f_\alpha(x) = \sum_{j \in \mathbb{Z}^n} (j - x)^\alpha \phi(x - j) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-x)^{\alpha - \beta} \phi *' (\ )^\beta.$$

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Hence f_α is a polynomial by the assumption. But f_α is 1-periodic, so f_α must be a constant. Thus the Fourier coefficients $\hat{f}_\alpha(j) = 0$ for all $j \in \mathbb{Z}^n \backslash \{0\}$. From the proof of Lemma 2.1 we see that

$$\hat{f}_\alpha(j) = \hat{\phi}_\alpha(2\pi j).$$

This together with (2.4) verifies the condition (2.6). Applying the Poisson summation formula to ϕ, we obtain

$$1 = \phi *' 1 = \hat{\phi}(0).$$

This verifies the condition (2.5). ∎

We mention that Theorem 2.2 was first proved by Strang and Fix in [20] for a compactly supported function. See [1, 5, 7, 13 for various interesting extensions of Strang and Fix's result.

In what follows, we consider the Strang–Fix conditions for several functions. Let $\phi := \{\phi_1, ..., \phi_N\}$ be a finite collection of functions from $\mathscr{L}_1^{k-1}$. We denote by $S_0(\Phi)$ the space of the finite linear combinations of functions in Φ and their integer translates. The collection Φ is said to satisfy the Strang–Fix conditions of order k provided there exists a function ϕ in $S_0(\Phi)$ satisfying the Strang–Fix conditions (2.5) and (2.6) of order k.

COROLLARY 2.3. *A finite collection Φ from $\mathscr{L}_1^{k-1}$ satisfies the Strang–Fix conditions of order k if and only if there exists a $\psi \in S_0(\Psi)$ such that the mapping $\psi *'$ is the identity on Π_{k-1}.*

Proof. The sufficiency is an immediate consequence of Theorem 2.2. To show the necessity we also apply Theorem 2.2. It follows that there is a function $\rho \in S_0(\Phi)$ such that the mapping $1 - \rho *'$ is degree-reducing on Π_{k-1}. It remains to show that there exists a finite linear combination ψ of ρ and its integer translates such that $\psi *'$ is the identity operator on Π_{k-1}. This was already proved in [14, Lemma 2.6] for the case where the functions in Φ satisfy the condition (1.1). That proof can be carried over verbatim for the present setting. ∎

3. Lower Bounds for Approximation Order

Let k be a positive integer, $1 \leqslant p \leqslant \infty$, and $\Phi = \{\phi_1, ..., \phi_N\}$ a finite subset of $\mathscr{L}_p^{k-1}$. We denote by $S(\Phi)$ the closure of $S_0(\Phi)$ in L_p. Furthermore, the h-scaling of $S(\Phi)$, $h > 0$, is denoted by $S_h(\Phi)$, namely,

$$S_h(\Phi) := \{\sigma_h g : g \in S(\Phi)\},$$

where σ_h is the scaling operator $\sigma_h: f \mapsto f(\cdot/h)$. In this section our goal is to estimate lower bounds for the approximation order provided by $\{S_h(\Phi): h > 0\}$. The main result is the following:

THEOREM 3.1. *Let Φ be a finite collection of functions in $\mathscr{L}_p^{k-1}$. If Φ satisfies the Strang–Fix conditions of order k, then for $1 \leqslant p < \infty$*

$$\operatorname{dist}(f, S_h(\Phi))_p := \inf\{\|f - g\|_p : g \in S_h(\Phi)\}$$
$$= o(h^{k-1}), \qquad \text{for all} \quad f \in W_p^{k-1}.$$

Proof. This theorem is a consequence of Corollary 2.3 and Theorem 3.4 that is proved later.

To handle L_p-approximation $(1 \leqslant p < \infty)$, we apply a smoothing technique as employed in [14]. Let χ be an element of $C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \chi \subset [-1, 1]^n$, $\chi \geqslant 0$ and $\int \chi = 1$. Set

$$\chi_h := \chi(\cdot/h)/h^n, \qquad h > 0.$$

For a given function $f \in L_p(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$ and $h > 0$, define

$$J_h f(x) := \int_{\mathbb{R}^n} (f - \nabla_u^k f)(x)\, \chi_h(u)\, du, \qquad x \in \mathbb{R}^n. \tag{3.1}$$

Clearly the operator J_h commutes with difference and differential operators. In the univariate case $(n = 1)$, such a smoothing technique was employed by DeVore [8] in studying degree of approximation. In the multivariate case, this scheme was used by Jia [12], and then developed in the recent work [14]. The following lemma is quoted from [14].

LEMMA 3.2. *For $f \in L_p(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$, the functions $f_h := J_h f$ are C^∞-smooth. Moreover, there exists a constant C depending only on k and n such that*

$$\left(\sum_{v \in \mathbb{Z}^n} |f_h(hv)|^p h^n \right)^{1/p} \leqslant C \|f\|_p$$

for $1 \leqslant p < \infty$.

The next lemma says that $J_h f = f_h$ is a good approximation of f when h is small.

LEMMA 3.3. *For all $1 \leqslant p < \infty$, $0 \leqslant m \leqslant k - 1$, and $f \in W_p^{k-1}$, we have*

$$|f - f_h|_{m,p} = o(h^{k-1-m}) \qquad \text{as} \quad h \to 0.$$

A proof for a similar result can be found in [14]. That proof with a simple modification may be used for the present case as well.

We are now in a position to complete the proof of Theorem 3.1. To this end, by virtue of Corollary 2.3, we need only prove the following:

THEOREM 3.4. *Let* $\phi \in \mathscr{L}_p^{k-1}$. *For* $f \in W_p^{k-1}(\mathbb{R}^n)$ $(1 \leqslant p < \infty)$ *and* $h > 0$, *set*

$$Q_h f(x) := \sum_{j \in \mathbb{Z}^n} f_h(hj)\,\phi(x/h - j), \qquad x \in \mathbb{R}^n,$$

where $f_h := J_h f$. *If* $\phi *'$ *is the identity on* Π_{k-1}, *then*

$$\| f - Q_h f \|_p = o(h^{k-1}) \qquad \text{as} \quad h \to 0.$$

Proof. By Lemma 3.3 we have

$$\| f - f_h \|_p = o(h^{k-1}).$$

Hence it remains to show that

$$\| f_h - Q_h f \|_p = o(h^{k-1}).$$

Observe that with $I := [0, 1)^n$, for any $g \in L_p(\mathbb{R}^n)$,

$$\| g \|_p^p = \sum_{\alpha \in \mathbb{Z}^n} \int_{h(I-\alpha)} |g(x)|^p\, dx = \int_{hI} \sum_{\alpha \in \mathbb{Z}^n} |g(x + h\alpha)|^p\, dx.$$

Taking g to be $f_h - Q_h f$ in the above, we obtain

$$\| f_h - Q_h f \|_p = \left(\int_{hI} \| a_x \|_{l_p}^p\, dx \right)^{1/p}, \tag{3.2}$$

where $a_x := \mathbb{Z}^n \to \mathbb{C}$ denotes the sequence given by the rule

$$\alpha \mapsto a_x(\alpha) := f_h(x + h\alpha) - Q_h f(x + h\alpha), \qquad \alpha \in \mathbb{Z}^n,$$

for each $h > 0$ and $x \in hI$.

Clearly we need to estimate the l_p-norm of the sequence a_x for $x \in hI$. To do this, we keep $h > 0$ and $x \in hI$ fixed for the moment. For each $\zeta \in \mathbb{R}^n$ and $g \in C^k(\mathbb{R}^n)$, let $\Lambda_\zeta g$ be the Taylor polynomial of g of degree $k-1$ about ζ. Since $\phi *'$ is the identity on Π_{k-1}, we have

$$p = \sum_{j \in \mathbb{Z}^n} p(hj)\,\phi(\cdot/h - j), \qquad \text{for all} \quad p \in \Pi_{k-1}.$$

Then it follows that

$$\begin{aligned} a_x(\alpha) &= (f_h - Q_h f)(x + h\alpha) \\ &= (\Lambda_{x+h\alpha} f_h - Q_h f)(x + h\alpha) \\ &= \sum_{j \in \mathbb{Z}^n} (\Lambda_{x+h\alpha} f_h - f_h)(hj)\, \phi(x/h + \alpha - j), \qquad \alpha \in \mathbb{Z}^n. \end{aligned}$$

After a change of variables $v := j - \alpha$, the above equation becomes

$$a_x(\alpha) = \sum_{v \in \mathbb{Z}^n} b_x(\alpha, v)\, \phi(x/h - v), \qquad \text{for} \quad \alpha \in \mathbb{Z}^n, \tag{3.3}$$

where

$$b_x(\alpha, v) := (\Lambda_{x+h\alpha} f_h - f_h)(hv + h\alpha), \qquad \alpha, v \in \mathbb{Z}^n.$$

Next, we estimate the l_p-norm of $b_x(\cdot, v)$. We first assume that $k \geqslant 2$. Observe that for every $\alpha \in \mathbb{Z}^n$, the Taylor polynomial of the function $\Lambda_{x+h\alpha} f_h - f_h$ of degree $k-2$ about $x + h\alpha$ is zero, and hence $b_x(\alpha, v)$ is equal to the corresponding remainder evaluated at the point $y = hv + h\alpha$. Then by Taylor's remainder formula,

$$b_x(\alpha, v) = \int_0^1 D_{hv-x}^{k-1} (\Lambda_{x+h\alpha} f_h - f_h)(x + h\alpha + t(hv - x)) \frac{(1-t)^{k-2}}{(k-2)!}\, dt.$$

It is easy to see that for any smooth function g and vectors $\xi, \zeta \in \mathbb{R}^n$, we have

$$D_\xi^{k-1} \Lambda_\zeta\, g(z) = D_\xi^{k-1} g(\zeta) \qquad \text{for all} \quad z \in \mathbb{R}^n.$$

Applying this formula to $g = f_h$, $\xi = hv - x$, and $\zeta = x + h\alpha$, we get

$$D_{hv-x}^{k-1} \Lambda_{x+h\alpha} f_h(x + h\alpha + t(hv - x)) = D_{hv-x}^{k-1} f_h(x + h\alpha).$$

Therefore

$$\begin{aligned} b_x(\alpha, v) &= \int_0^1 D_{hv-x}^{k-1} (f_h(x + h\alpha) - f_h(x + h\alpha + t(hv - x))) \frac{(1-t)^{k-2}}{(k-2)!}\, dt \\ &= \int_0^1 D_{hv-x}^{k-1} \nabla_{t(x-hv)} f_h(x + h\alpha) \frac{(1-t)^{k-2}}{(k-2)!}\, dt. \end{aligned}$$

Furthermore, since the operator J_h commutes with difference and differential operators, we see that

$$b_x(\alpha, v) = \int_0^1 F_h(h\alpha, v, t)\, dt, \qquad \alpha, v \in \mathbb{Z}^n, \tag{3.4}$$

where $F_h := J_h F$ with

$$F(\cdot, v, t) := D^{k-1}_{hv-x} \nabla_{t(x-hv)} f(x+\cdot) \frac{(1-t)^{k-2}}{(k-2)!}. \tag{2.5}$$

Applying the generalized Minkowski inequality yields

$$\|b_x(\cdot, v)\|_{l_p} \leqslant \int_0^1 \|F_h(h\cdot, v, t)_|\|_{l_p}\, dt, \qquad v \in \mathbb{Z}^n,$$

where $g_|$ denotes the restriction of a continuous function g to $\mathbb{Z}^n$. By Lemma 3.2 we have

$$\|F_h(h\cdot, v, t)_|\|_{l_p} \leqslant C_1 h^{-n/p} \|F(\cdot, v, t)\|_p,$$

for every $v \in \mathbb{Z}^n$ and $t \in [0, 1]$, where C_1 is a constant depending only on k and n. It follows from (3.5) that there exists a constant C_2 dependent only on k and n such that

$$\begin{aligned}\|F(\cdot, t, v)\|_p &\leqslant C_2 \|hv - x\|^{k-1} \sum_{|\gamma| = k-1} \|\nabla_{t(x-hv)} D^\gamma f\|_p \\ &\leqslant C_2 \|hv - x\|^{k-1} \omega_k(f, \|x - hv\|)_p,\end{aligned}$$

for every $v \in \mathbb{Z}^n$ and $t \in [0, 1]$, where ω_k stands for the modulus of smoothness defined by

$$\omega_m(g, \tau)_p := \sup_{\|z\| \leqslant \tau} \sum_{|\gamma| = m-1} \|D^\gamma \nabla_z g\|_p,$$

for $g \in W^m_p(\mathbb{R}^n)$ and a positive integer m. Then it follows that

$$\|b_x(\cdot, v)\|_{l_p} \leqslant C h^{k-1-n/p} \omega_k(f, \|x - hv\|)_p \|x/h - v\|^{k-1}, \qquad v \in \mathbb{Z}^n, x \in [0, h]^n \tag{3.6}$$

for some constant C dependent only on k and n. Although (3.6) was proved for the case $k \geqslant 2$, it also holds for $k = 1$. For, if $k = 1$,

$$b_x(\alpha, v) = f_h(x + h\alpha) - f_h(hv + h\alpha),$$

from which (3.6) follows.

By virtue of (3.6), applying the Minkowski inequality to (3.3), we obtain

$$\|a_x\|_{l_p} \leqslant C h^{k-1-n/p} \sum_{v \in \mathbb{Z}^n} \omega_k(f, \|x - hv\|)_p |\psi(x/h - v)|, \tag{3.7}$$

where

$$\psi := (1 + \|\cdot\|)^{k-1} \phi. \tag{3.8}$$

For each positive integer M, the sum $\sum_{v \in \mathbb{Z}^n}$ can be split into two sums: $\sum_{\|v\| < M}$ and $\sum_{\|v\| \geq M}$. We use the inequality

$$\omega_k(f, \|x - hv\|)_p \leq \omega_k(f, hM)_p$$

for the case $\|v\| < M$ and use the estimate

$$\omega_k(f, \|x - hv\|)_p \leq 2 \, |f|_{k-1, p}$$

for the case $\|v\| \geq M$. Then (3.7) yields

$$\|a_x\|_{l_p} \leq Ch^{k-1-n/p}(\omega_k(f, hM)_p \, |\psi^\circ(x/h)| + 2 \, |f|_{k-1, p} \, |\psi^M(x/h)|), \tag{3.9}$$

where ψ° is given by (1.2), and

$$\psi^M := \sum_{\|j\| \geq M} |\psi(\cdot - j)|. \tag{3.10}$$

In the rest of the proof we estimate the quantity $(\int_{hI} \|a_x\|_{l_p}^p \, dx)^{1/p}$. It is clear that $\|\psi^\circ(\cdot/h)\|_p \, (hI) = h^{n/p} \, |\psi|_p$ and $\|\psi^M(\cdot/h)\|_p \, (hI) = h^{n/p} \, \|\psi^M\|_p \, (I)$. This, together with (3.9), yields

$$\left(\int_{hI} \|a_x\|_{l_p}^p \, dx\right)^{1/p} \leq Ch^{k-1}\Bigg(\sum_{|\gamma| = k-1} \omega_k(D^\gamma f, hM)_p \, |\psi|_p$$

$$+ 2 \, |f|_{k-1, p} \, \|\psi^M\|_p \, (I)\Bigg), \tag{3.11}$$

for all positive integers M. In view of (3.8) and the fact that $\phi \in \mathscr{L}_p^{k-1}$, we *conclude that* $\psi \in \mathscr{L}_p$. Hence $|\psi|_p < \infty$ *and* $\|\psi^M\|_p \, (I) \to 0$ *as* $M \to \infty$. It follows from (3.11) that

$$\int_{hI} (\|a_x\|_{l_p}^p \, dx)^{1/p} \, dx = o(h^{k-1}),$$

if we choose M to be the integer part of $1/\sqrt{h}$. The theorem is proved. ∎

Remark 3.5. In a similar way, we can show that the estimate given in Theorem 3.1 is also valid for $p = \infty$ if, in addition, the function f to be approximated is in $W_\infty^{k-1} \cap C^{k-1}$, and the functions $\phi_s \in \Phi_s$ satisfies that $\|\psi_s^M\|_\infty \, ([0, 1)^n) \to 0$ as $M \to \infty$, $s = 1, ..., N$, where ψ_s and ψ_s^M are given in the same fashion as in (3.8) and (3.10), respectively.

4. THE LOCAL-CONTROLLED APPROXIMATION ORDER

From [11] we see that the converse of Theorem 3.1 is not true in general. However, as was mentioned before, Strang and Fix [20] succeeded in characterizing the controlled approximation order provided by one compactly supported function and de Boor and Jia [3] in characterizing the local approximation order provided by finitely many compactly supported functions. In what follows, we introduce the notion of local-controlled approximation, which appeared in [7] in a different version and terminology. In this section we characterize the local-controlled approximation of order $o(h^{k-1})$. The authors of [10] considered local-controlled L_∞-pproximation in a different version, by giving a characterization result, in which the approximation order $\mathcal{O}(h^k)$ is required.

Let k be a positive integer, $1 \leqslant p \leqslant \infty$, and let $\Phi = \{\phi_1, ..., \phi_N\}$ be a finite collection of functions in $\mathcal{L}_p^{k-1}$. We say that Φ provides local-controlled L_p-approximation of order $o(h^{k-1})$ if each $f \in W_p^{k-1}$ there exist sequences $c_s^h \in l_p$, $h > 0$, $s = 1, ..., N$, such that

(i) $\|f - \sigma_h(\sum_{s=1}^N (\phi_s *' c_s^h))\, h^{-n/p}\|_p = o(h^{k-1})$, as $h \to 0$,

(ii) $\|c_s^h\|_{l_p} \leqslant C\, \|f\|_p$, $s = 1, ..., N$, and

(iii) $\operatorname{dist}(hv, \operatorname{supp} f) > r \Rightarrow c_s^h(v) = 0$, $s = 1, ..., N$,

for some constants C and r independent h.

THEOREM 4.1. *Let $\Phi = \{\phi_1, ..., \phi_N\}$ be a finite collection of functions in $\mathcal{L}_p^{k-1}$, where $1 \leqslant p < \infty$. Then Φ provides local-controlled L_p-approximation of order $o(h^{k-1})$ if and only if it satisfies the Strang–Fix conditions of order k; this characterization is also true for the case $p = \infty$, if the conditions in Remark* 3.5 *are satisfied.*

Proof. It remains to prove the necessity. Along the line of [6], this was proved in [14] for the case when the following stronger conditions are satisfied: the approximation order is at least $\mathcal{O}(h^k)$ and the functions in Φ satisfy the decay condition (1.1). Some modification of that proof has to be made for the present case. We thus give the outline of the proof for the necessity.

We approximate a tensor product of univariate B-splines—namely, the function

$$u(x) := \prod_{l=1}^{n} M_{k+1}(x_l), \qquad x = (x_1, ..., x_n) \in \mathbb{R}^n,$$

where M_{k+1} is the well-known B-spline whose Fourier transform is given by

$$\hat{M}_{k+1}(t)=\left(\frac{\sin(t/2)}{t/2}\right)^{k+1}. \tag{4.1}$$

Since $u \in W_p^k(\mathbb{R}^n)$, we can find sequences c_s^h $(s=1, ..., N, h>0)$ so that the conditions (i), (ii), and (iii) are satisfied. Let

$$u_h := h^{-n/p}\sigma_h\left(\sum_{s=1}^{N} \phi_s *' c_s\right)$$

and set $g_h := u - u_h$. Then the property (i) implies that

$$\|g_h\|_p = o(h^{k-1}) \qquad \text{as} \quad h \to 0.$$

We claim that

$$\|D^\alpha \hat{g}_h\|_\infty = o(h^{k-1}) \qquad \text{as} \quad h \to 0 \text{ for } |\alpha| = m \leqslant k-1. \tag{4.2}$$

To this end we apply the differential operator D^α to $\hat{g}_h$ and obtain

$$D^\alpha \hat{g}_h(\xi) = \int_{\mathbb{R}^n} (-ix)^\alpha\, g_h(x)\, e^{-i\xi\cdot x}\, dx.$$

It follows that

$$\|D^\alpha \hat{g}_h\|_\infty \leqslant \int_{\mathbb{R}^n} \|x\|^m\, |g_h(x)|\, dx =: J \qquad \text{for} \quad |\alpha| = m.$$

Let $R := r + k$. Since the support of u is included in the ball $\{x \in \mathbb{R}^n\colon \|x\| \leqslant k\}$, by the property (iii) we have

$$c_s^h(v) = 0 \qquad \text{for} \quad \|vh\| > R. \tag{4.3}$$

Write

$$J = \int_{\|x\|<2R} \|x\|^m\, |g_h(x)|\, dx + \int_{\|x\|\geqslant 2R} \|x\|^m\, |g_h(x)|\, dx =: J_1 + J_2.$$

Applying Hölder's inequality to the first integral J_1, we obtain

$$J_1 \leqslant \left(\int_{\|x\|<2R} (\|x\|^m)^q\, dx\right)^{1/q} \left(\int_{\|x\|<2R} |g_h(x)|^p\, dx\right)^{1/p} = o(h^{k-1}),$$

where q is the exponential conjugate to p, i.e., $1/p + 1/q = 1$.

Let us next estimate J_2. The fact that $u(x)=0$ for $\|x\| \geqslant R$ and (4.3) yield

$$g_h(x) = \sum_{s=1}^{N} \sum_{\|hv\| \leqslant R} h^{-n/p} c_s^h(v)\, \phi_s(x/h - v), \qquad \text{for} \quad \|x\| \geqslant 2R.$$

Consequently,

$$\begin{aligned} J_2 &\leqslant \int_{\|x\| \geqslant 2R} \|x\|^{k-1}\, |g_h(x)|\, dx \\ &\leqslant h^{-n/p} \sum_{s=1}^{N} \sum_{\|hv\| \leqslant R} |c_s^h(v)| \int_{\|x\| \geqslant 2R} \|x\|^{k-1}\, |\phi_s(x/h - v)|\, dx \\ &= h^{k-1+n/q} \sum_{s=1}^{N} \sum_{\|hv\| \leqslant R} |c_s^h(v)| \\ &\quad \times \int_{\|y+v\| > 2R/h} \|y+v\|^{k-1}\, |\phi_s(y)|\, dy, \end{aligned} \tag{4.4}$$

where we have made the change of variables $y = x/h - v$ in the last equality. To estimate the above integral we note that the inequalities $\|y\|/2 \leqslant \|y+v\| \leqslant 2\,\|y\|$ hold, if $\|y+v\| \geqslant 2R/h$ and $\|v\| \leqslant R/h$. Then

$$\begin{aligned} \int_{\|y+v\| \geqslant 2R/h} \|y+v\|^{k-1}\, |\phi_s(y)|\, dy &\leqslant \int_{\|y\| \geqslant R/h} (2\,\|y\|)^{k-1}\, |\phi_s(y)|\, dy \\ &= 2^{k-1} \int_{\|y\| \geqslant R/h} |\psi_s(y)|\, dy \leqslant \mu_h, \end{aligned}$$

where $\psi_s := (1 + \|\cdot\|)^{k-1}\, \phi_s$ and

$$\mu_h := 2^{k-1} \max_{1 \leqslant s \leqslant N} \int_{\|y\| \geqslant R/h} |\psi_s(y)|\, dy.$$

It follows that

$$J_2 \leqslant h^{k-1+n/q} \mu_h \sum_{s=1}^{N} \sum_{\|hv\| \leqslant R} |c_s^h(v)|. \tag{4.5}$$

Since c_s^h satisfy the condition (ii), by Hölder's inequality we have

$$\sum_{\|hv\| \leqslant R} |c_s^h(v)| \leqslant \Bigg(\sum_{\|hv\| \leqslant R} |c_s^h(v)|^p \Bigg)^{1/p} \Bigg(\sum_{\|hv\| \leqslant R} 1 \Bigg)^{1/q} \leqslant C\, \|u\|_p\, h^{-n/q}.$$

Combining this with (4.5), we get

$$J_2 \leqslant C h^{k-1} \mu_h$$

for some constant C independent of h. Since $\psi_s \in \mathscr{L}_p \subset \mathscr{L}_1 = L_1$, $s = 1, ..., N$, we have $\int_{R/h} |\psi_s(y)|\, dy \to 0$; hence $\mu_h \to 0$ as $h \to 0$. Therefore $J_2 = o(h^{k-1})$ follows. To summarize, we have proved our claim (4.2).

It follows from (4.1) that

$$\hat{u}(\xi) = \prod_{l=1}^{n} \left(\frac{\sin(\xi_l/2)}{\xi_l/2}\right)^{k+1}, \qquad \xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n.$$

Hence

$$\hat{u}(0) = 1 \tag{4.6}$$

and

$$\lim_{h \to 0} D^\alpha \hat{u}(\xi/h)/h^{k-1} = 0 \qquad \text{for} \quad \xi \in \mathbb{R}^n \backslash \{0\} \text{ and } |\alpha| < k. \tag{4.7}$$

Recall that $g_h = u - u_h$. Thus (4.2), (4.6), and (4.7) together yield

$$\lim_{h \to 0} \hat{u}_h(0) = 1 \tag{4.8}$$

and

$$\lim_{h \to 0} D^\alpha \hat{u}_h(\xi/h)/h^{k-1} = 0 \qquad \text{for all} \quad \xi \in \mathbb{R}^n \backslash \{0\} \text{ and } |\alpha| < k. \tag{4.9}$$

It remains to show that the properties (4.8) and (4.9) imply the Strang–Fix conditions of order k. A proof for this implication can be found in Ref. [15]. The proof of the theorem is thus complete. ∎

It is clear that the approximation scheme given in Theorem 3.4 satisfies both the conditions (ii) and (iii). Therefore the combination of Theorems 3.1 and 4.1 gives a characterization of the local-controlled L_p-approximation order $o(h^{k-1})$ provided by finitely many functions from $\mathscr{L}_p^{k-1}$.

Remark. After completing this work, we learned that de Boor *et al.* obtained in [2] a complete characterization of closed shift-invariant spaces of $L_2(\mathbb{R}^n)$, which provide L_2-approximation of order k or density order $k-1$. The concept of approximation order $o(h^{k-1})$ appearing in the present paper coincides with that of density order $k-1$ introduced in [2].

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